SOLUTIONS OF EXERCISES.

166

A CIRCLE cuts a parabola, and the centroid of the four points of intersection is found. What is the locus of the centre of the circle if this point be fixed?

[*W. M. Thornton.*]

SOLUTION.

The equations to the curves are

$$y^2 = 2px, \quad (x - u)^2 + (y - v)^2 = r^2.$$

At the points of intersection

$$[(x - u)^2 - r^2 + 2px + v^2]^2 = 8pv^2x,$$

or

$$x^4 + 4(p - u)x^3 + \ . \ . \ . = 0.$$

By hypothesis, $x_1 + x_2 + x_3 + x_4$ is constant; or

$$p - u = c,$$

which is a straight line at right angles to the axis of the parabola.

[*R. H. Graves.*]

167

IF $y_1y_2 = k^2$ (a constant), where (x_1, y_1) and (x_2, y_2) are points on the parabola $y^2 = 4ax$, find the locus of the intersections of the normals at these points.

[*R. H. Graves.*]

SOLUTION.

If (u, v) be the foot of the normal from (x, y) to $y^2 = 2px$

$$v^3 + 2p(p - x)v - 2p^2y = 0.$$

By hypothesis $v_1v_2 = k^2$. But $v_0v_1v_2 = 2p^2y$. Therefore

$$v_0 = \frac{2p^2y}{k^2},$$

whence we find for the equation to the required locus

$$4p^4y^2 = 2pk^4(x - p) + k^6.$$

[*W. M. Thornton, and others.*]

205

A TRIANGLE PQR is inscribed in an ellipse having the two sides PQ and PR passing through the foci, and the line QR produced meets the tangent at P at the point S. The polar of S with respect to a concentric circle through P passes through the centre of curvature of the ellipse at the point P.

[*R. H. Graves.*]

SOLUTION.

Let
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

be the equation to the ellipse, and let the co-ordinates of the points P and S be (x', y') and (x_1, y_1).

Then, S is the pole of the normal at P. (See solution to Exercise 139.)

$$\therefore x_1=\frac{a^4}{(a^2-b^2)x'} \text{ and } y_1=-\frac{b^4}{(a^2-b^2)y'}.$$

The centre of curvature is given by the equations

$$\alpha=\frac{(a^2-b^2)x'^3}{a^4} \text{ and } \beta=-\frac{(a^2-b^2)y'^3}{b^4};$$

$$\therefore \alpha x_1+\beta y_1=x'^2+y'^2,$$

and the truth of the proposition follows. [*R. H. Graves.*]

206

IF from any point four normals be drawn to each of the hyperbolas,

$$x^2-y^2=a^2 \quad\text{and}\quad xy=k^2,$$

the centres of mean position of the feet of the two sets of normals are coincident.

[*R. H. Graves.*]

SOLUTION.

The normal from (x', y') to $x^2-y^2=a^2$ at (x, y) is

$$xy'+x'y=2xy;$$

$$\therefore (x^2-a^2)(4x^2-4xx'+x'^2)=x^2y'^2;$$

$$\therefore x_1+x_2+x_3+x_4=x'.$$

A like result will be found for $xy=k^2$.

214

AN equilateral hyperbola is circumscribed to a triangle. Find the greatest and least of the values of the transverse axis. [*R. H. Graves.*]

SOLUTION.

Since the curve passes through the orthocentre, the minimum is zero, and the maximum is two-thirds of the altitude. [*R. H. Graves.*]

217

A CIRCLE is drawn at random within a given circle. What is the probability that the random circle contains the centre of the given circle?

[*Artemas Martin.*]

SOLUTION.

Let A be the centre of the given circle, and P the centre of the random circle; through P draw the radius OB of the given circle; then OP is the distance between their centres. Put $r=$ radius of the given circle, and let $OP=x$. In order that the random circle may contain the centre of the given circle, OP, the distance between their centres must be less than $\frac{1}{2}r$. While the centre of the random circle is at P, its radius can have any value from 0 to $r-x$; but it will not contain the centre of the given circle unless its radius exceeds x; hence in order to include the centre of the given circle, the radius of the random circle can have any value from x to $r-x$; therefore for an assigned value of x the probability that the random circle contains the centre of the given circle is

$$p_x = \frac{r-2x}{r-x}.$$

But x may have any value from 0 to $\frac{1}{2}r$, and all these values are equally probable. P will be at the distance x from the centre of the given circle if it is anywhere on the circumference of the circle whose centre is at O and radius x; therefore the probability required is

$$p = \int_0^{\frac{1}{2}r} p_x \times 2\pi x dx \div \int_0^r 2\pi x dx,$$

$$= \frac{2}{r^2}\int_0^{\frac{1}{2}r} x\left(\frac{r-2x}{r-x}\right) dx,$$

$$= \frac{2}{r^2}\int_0^{\frac{1}{2}r}\left(2x+r-\frac{r^2}{r-x}\right) dx,$$

$$= 2(\tfrac{3}{4} - \log_e 2).$$ [*Artemas Martin.*]